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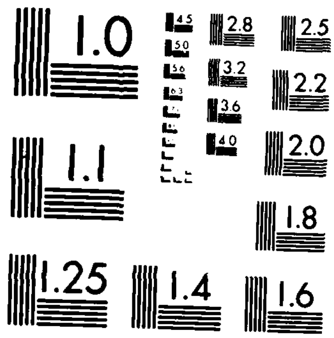

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JOINT MOMENTS OF THE MAXIMUM
IN A CRITICAL BRANCHING PROCESS

BY

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1. INTRODUCTION

Let $\{Z_n\}$, $n \geq 0$, with $Z_o = 1$ denote a critical Galton-Watson branching process, and let the σ-field $F_k \equiv \sigma(Z_1, Z_2, \ldots Z_k)$ for $k \geq 1$. Denote

$$M_n = \max_{1 \leq \ell \leq n} Z_\ell. \tag{1.1}$$

(Kämmerle and Schuh [2] pp. 601-602).

Assume $E\, M_n^r < \infty$ for all $r \geq 1$, $n \geq 1$.

Moments of M_n have been studied by a number of authors (See Athreya [1], Kämmerle and Schuh [2], Pakes [3], Weiner [5]). See Kämmerle and Schuh for further background on this problem and other references. In this paper the asymptotic order of magnitude of joint moments of M_k, M_n for $n >> k >> 1$ are studied. Much use is made of the results of Athreya [1], Kämmerle and Schuh [2] to obtain the asymptotic orders of magnitude, and relevant results are given in the next section.

2. PRIOR RESULTS AND NOTATION

The results and notation (2.1) - (2.4) will be used assuming $Z_0 = 1$: (Kämmerle and Schuh [2], p. 602).

$$\{Z_n, F_n\};\ n \geq 0 \text{ is a martingale} \tag{2.1}$$

$$EZ_n = 1,\quad \text{Var } Z_n = n\sigma^2,\ n \geq 1 \tag{2.2a}$$

$$P(Z_n = 0) \longrightarrow 1 \text{ as } n \longrightarrow \infty \tag{2.2b}$$

The following notation will be used in this paper:

for $a_n, b_n > 0$,

$$a_n \sim b_n \Leftrightarrow \tag{2.3}$$

$\underline{\lim}_{n \to \infty} a_n/b_n > a > 0$ and $\overline{\lim}_{n \to \infty} a_n/b_n < b < \infty$ where $a,b > 0$ are constants.

$$a \leq \overline{\lim}_{n \to \infty} a_n \leq b \Leftrightarrow \tag{2.4}$$

$$a \leq \underline{\lim}\, a_n \leq \overline{\lim}\, a_n \leq b$$

The following result is critical for the argument:

If $E(Z_1^2) < \infty$ then if $Z_0 = \ell$,

$$E(M_n | Z_0 = \ell)/\log n \longrightarrow \ell \text{ as } n \longrightarrow \infty \tag{2.5}$$

(Athreya [1], p. 1).

We note that it was earlier shown (Kämmerle and Schuh [2], p.603) that for $Z_0 = 1$, $\frac{1}{e} \leq \overline{\lim}_{n \to \infty} E(M_n)/\log n \leq 1$.

See also (Pakes [3], Weiner [5]) for earlier results.

The other results crucial to the argument are as follows (Kämmerle and Schuh [2], pp. 607-610):

If $Z_0 = k \geq 1$ and $E(Z_n^r | Z_0 = k) < \infty$ all $r \geq 1$, $n \geq 1$, then for $r \geq 2$,

$$\lim_{n \to \infty} E(Z_n^r | Z_0 = k)/n^{r-1} = \left(\frac{\sigma^2}{2}\right)^{r-1} r! \tag{2.6}$$

and also for $k \geq 1$.

$$k \left(\frac{\sigma^2}{2}\right)^{r-1} r! \leq \overline{\lim_{n\to\infty}} E(M_n^r | Z_0 = k)/n^{r-1} \leq k\left(\frac{r}{r-1}\right)^r \left(\frac{\sigma^2}{2}\right)^{r-1} r! \qquad (2.7)$$

LEMMA 1. For integers $r, s \geq 1, \ell \geq 1$,

$$\ell \left(\frac{\sigma^2}{2}\right)^{r+s-1} (r+s)! \leq E(M_n^r Z_n^s | Z_0 = \ell)/n^{r+s-1} \leq \ell\left(\frac{r+s}{r+s-1}\right)^{r+s} \left(\frac{\sigma^2}{2}\right)^{r+s-1} (r+s)! \qquad (2.8)$$

PROOF.

$E(Z_n^{r+s} | Z_0 = \ell) \leq E(M_n^r Z_n^s | Z_0 = \ell) \leq E(M_n^{r+s} | Z_0 = \ell)$.

The result follows immediately from (2.6), (2.7). □

LEMMA 2. For $1 \leq k < n$, $r, s \geq 1$

$$E(Z_k^r Z_n^s) \geq E(Z_k^{r+s}) \qquad (2.9i)$$

$$E(Z_k^r Z_n) = E(Z_k^{r+1}) \qquad (2.9ii)$$

$$E(M_k^r Z_n) = E(M_k^r Z_k) \qquad (2.9iii)$$

PROOF. The results follow by similar arguments and only the first result will be indicated.

$$E(Z_k^r Z_n^s) = E\,E(Z_k^r Z_n^s | F_k) = E(Z_k^r E(Z_n^s | F_k)) \geq E(Z_k^{r+s})$$

since $E(Z_k^r Z_n^\ell)$ $\{Z_n\}$ is a martingale and by Jensen's Conditional inequality (Rao [4] p. 110). □

3. JOINT MOMENTS

THEOREM 1. For $1 \le k < n$, $r \ge 1$, $Z_0 = 1$,

$$\left(\frac{\sigma^2}{2}\right)^r (r+1)! \le \varliminf_{\substack{k\to\infty \\ n-k\to\infty}} E(M_k^r M_n)/(k^r \log(n-k)) \le \left(\frac{r+1}{r}\right)^{r+1} \left(\frac{\sigma^2}{2}\right)^r (r+1)! \quad (3.1)$$

PROOF.

$$E\, M_k^r M_n = E\, E\,(M_k^r \; M_n | F_k) = E(M_k^r\, E\, M_n | F_k) \sim$$

$E(M_k^r\,(Z_k I(Z_k \ge 1) \log(n-k) + E(M_k^r\, M_k I(Z_k = 0))$ by (2.5) where $I(A)$ is the indicator of A. For $n-k \gg 1$, $k \gg 1$

$$E(M_k^r M_n) \sim \log(n-k)\, E(M_k^r Z_k) + E(M_k^{r+1} I(Z_k = 0). \quad (3.2)$$

From

$$0 \le E(M_k^{r+1} I(Z_k = 0)) \le E(M_k^{r+1}). \quad (3.3)$$

using $n-k \to \infty$ and $Z_0 = 1$, then (3.2), (3.3), and (2.8) yield the result.□

THEOREM 2. For $1 \le k < n$, $r \ge 1$, $s \ge 2$,

$$\left(\frac{\sigma^2}{2}\right)^{r+s-1} s!\,(r+1)! \le \varlimsup_{\substack{k\to\infty \\ n-k\to\infty}} E(M_k^r M_n^s/k^r (n-k)^{s-1} \le \left(\frac{s}{s-1}\right)^s \left(\frac{r+1}{r}\right)^{r+1} \left(\frac{\sigma^2}{2}\right)^{r+s-1} (r+1)! \quad (3.4)$$

PROOF. For $n-k \gg 1$, $k \gg 1$, $Z_0 = 1$,

$$E(M_k^r M_n^s) = E(M_k^r M_n^s | F_k) = E(M_k^r E(M_n^s | F_k)) \sim E(M_k^r Z_k I(Z_k \ge 1)(n-k)^{s-1} + E(M_k^{r+1} I(Z_k = 0))$$

$$\sim E(M_k^r\, Z_k)(n-k)^{s-1} + E(M_k^{r+1}) \sim (n-k)^{s-1}\, E(M_k^r\, Z_k),$$ by (2.7), (2.2b).

Then (2.7), (2.8) yields the result. □

4. HIGHER ORDER JOINT MOMENTS

An example of possible higher-order asymptotic joint moment behavior is given. For ease of exposition, only orders of magnitude but not the explicit constants, involved in upper and lower bounds, are given.

THEOREM 3. For $Z_0 = 1$, $t \geq 1$, $s \geq 1$, $k < r < 1$ and $k \gg 1$, $n-r \gg 1$, $r-k \gg 1$ and in addition

$r - k \gg k$, then

$$E(M_k^s M_r^t M_n) \sim (k)^{s+1} (r-k)^t \log(n-r). \tag{4.1}$$

PROOF. For $k < r < n$, using (3.1)

$$E(M_k^s M_r^t M_n) = E(M_k^s E(M_r^t M_n | F_k)) \sim E(M_k^s Z_k I(Z_k \geq 1)) (r-k)^t \log(n-r)) + E(M_k^{s+t+1} I(Z_k = 0) \tag{4.2}$$

Since $E(M_k^s Z_k I(Z_k \geq 1)) = EM_k^s Z_k$.

then by (2.2b)

$$E(M_k^s M_r^t M_n) \sim (r - k)^t \log(n - r) E(M_k^s Z_k) + E(M_k^{s+t+1}). \tag{4.3}$$

By (2.7), (2.8) applied to (4.3),

$$E(M_k^s M_r^t M_n) \sim k^s (r-k)^t \log(n-r) + k^{s+t}. \tag{4.4}$$

Using the hypothesis that $r-k \gg k$, the result follows. □

THEOREM 4. For $Z_0 = 1$, $t \geq 1$, $s \geq 1$, $\ell \geq 2$ and $k < r < n$ with $k >> 1$, $n-r >> 1$, $r - k >> 1$ and in addition

$n-r >> k$, $r-k >> k$

then

$$E(M_k^s M_r^t M_n^\ell) \sim k^s(n-r)^{\ell-1}(r-k)^t \tag{4.5}$$

PROOF. For $k < r < n$, $k >> 1$, $n - r >> 1$, $r - k >> 1$,

$$\begin{aligned} E(M_k^s M_r^t M_n^\ell) &= E\,(M_k^s E(M_r^t M_n^\ell \mid F_k)) \sim E(M_k^s Z_k I(Z_k \geq 1))(n-r)^{\ell-1}(r-k)^t \\ &\quad + E(M_k{}^{s+t+\ell} I(Z_k=0)) \\ &\sim E(M_k^s Z_k)\,(n-r)^{\ell-1}(r-k)^t + E(M_k^{s+t+\ell}), \end{aligned} \tag{4.6}$$

by (2.2b) and (3.4).

By (2.7), (2.8) applied to (4.6), $E(M_k^s M_r^t M_n^\ell) \sim k^s(n-r)^{\ell-1}(r-k)^t + k^{s+t+1}$. (4.7)

Applying $n-r >> k$, $r-k >> k$ to (4.7) yields the result. □

REFERENCES

1. K. Athreya, On the Maximum Sequence in a Critical Branching Process, Statistical Laboratory Preprint 86-29, Dept. Mathematics and Statistics, Iowa State U., Ames, IA 50011, 1986.

2. K. Kämmerle and H.-J. Schuh, The Maximum in Critical Galton-Watson and Birth and Death Processes, Journal of Applied Probability, Vol. 23, pp. 603-613, 1986.

3. A. Pakes, Remarks on the Maxima of a Simple Critical Branching Process. Preprint, Journal of Applied Probability (to appear) 1987.

4. M.M. Rao, Probability Theory with Applications, Academic Press, Orlando,Florida, 1984.

5. H. Weiner, Moments of the Maximum in a Critical Branching Process, Journal of Applied Probability, Vol. 21, pp. 920-923, 1984.

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The asymptotic order of magnitude of the joint moments of the maxima in a critical Galton Watson process are given.

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